

A COHOMOLOGICAL CRITERION FOR p -SOLVABILITY

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ABSTRACT. Let G be a finite group, p a prime and P a Sylow p -subgroup of G . In this note we give a cohomological criterion for the p -solvability of G depending on the cohomology in degree 1 with coefficients in \mathbb{F}_p of both the normal subgroups of G and P . As a byproduct we bound the minimal number of quotients of order a power of p appearing in any normal series of G by the number of generators of P .

1. INTRODUCTION

Let G be a finite group and p an arbitrary fixed prime integer. We denote the i -th cohomology and i -th homology groups of G with coefficients in the field \mathbb{F}_p of p elements by $H^i(G) = H^i(G, \mathbb{F}_p)$ and $H_i(G) = H_i(G, \mathbb{F}_p)$ respectively, where the action considered is the trivial one. The first cohomology group is naturally isomorphic to the group of homomorphisms from G into \mathbb{F}_p , that is

$$H^1(G) \cong \text{Hom}(G, \mathbb{F}_p),$$

and

$$H_1(G) \cong H^1(G)^* \cong G/G^p[G, G],$$

where A^* denotes the dual group of an abelian group A , and G^p is the subgroup of G generated by the p -powers of the elements of G .

Suppose now that P is a Sylow p -subgroup of G . Since P has index in G coprime to p , the restriction map from $H^1(G)$ into $H^1(P)$ defines an injective group homomorphism, and it is the content of Tate's p -nilpotency criterion [4] that this injection is a group isomorphism if and only if G has a normal p -complement. In this note, we present a cohomological criterion for a finite group to be p -solvable based on Tate's characterization. In order to state our main result, we first need to introduce some notation. If $N \trianglelefteq G$ and $K \leq G$, then K acts in a natural way into $H^1(N)$, and we denote by $H^1(N)^K$ the subgroup of fixed points of this action. Let $O^p(G)$ be the smallest normal subgroup of G such that $G/O^p(G)$ is a p -group, so $H^1(G) \cong H^1(G/O^p(G))$. Similarly, $O^{p'}(G)$ is defined as the minimal normal subgroup of G such that $G/O^{p'}(G)$ has order coprime to p .

Theorem A. *Let G be a finite group, p a prime number and $P \in \text{Syl}_p(G)$. Write $M_1 = O^{p'}(G)$ and $M_i = O^{p'}(O^p(M_{i-1}))$ for $i \geq 2$. Then the following two conditions are equivalent:*

- (1) G is p -solvable.

2010 *Mathematics Subject Classification.* 20D10.

Key words and phrases. Cohomology, p -solvability criterion, p -length.

The first author was supported by grants MTM2011-28229- C02-01 and MTM2014-53810-C2-2-P, from the Spanish Ministry of Economy and Competitivity, the Ramon y Cajal Programme of the Spanish Ministry of Science and Innovation, grant RYC-2011-08885, and by the Basque Government, grants IT753-13 and IT974-16.

Second author is supported by grants MTM2011-28229- C02-01 and MTM2014-53810-C2-2-P, from the Spanish Ministry of Economy and Competitivity, and by Prometeo II/Generalitat Valenciana.

$$(2) \ H^1(P) \cong \bigoplus_{i \in \mathbb{N}} H^1(M_i)^P.$$

Observe that it is clear that condition (2) in Theorem A is equivalent to saying that the groups claimed to be isomorphic have the same dimension as \mathbb{F}_p -vector spaces.

Recall that the p -length in a p -solvable group is defined as the minimal number of quotients of order a power of p appearing in any normal series of the group. Note that an immediate consequence of Theorem A is the well-known fact that if G is p -solvable, then the number of generators of a Sylow p -subgroup P of G is greater than or equal to the p -length of G (see Huppert's Hauptsatz 6.6 in [1]). We may extend the definition of p -length to any finite group G , by saying that the **p -length** of G is equal to the minimal number of quotients of order a power of p appearing in any normal series of G . Then we can generalize Huppert's result as follows:

Theorem B. *Let G be a finite group, p a prime and $P \in \text{Syl}_p(G)$. Suppose that d is the number of generators of P , and that l is the p -length of G . Then $l \leq d$.*

Another result of Huppert states that for p and odd prime, the number of non- p -solvable chief factors of a finite group is bounded by the number of generators of a Sylow p -subgroup of the group (see Satz 2.3 of [2]). We will call the number of such chief factors the **non- p -solvable length** of the group, noting that this definition is slightly different from the non- p -solvable length introduced by E. I. Khukhro and P. Shumyatsky [3]. Consider a normal series of a group G whose quotients are either non- p -solvable chief factors, p -groups or p' -groups. Then we define the **generalized p -length** of G as the smallest possible number of quotients of order divisible by p appearing in such a series. It is then clear that the generalized p -length of a finite group is bounded by twice the number d of generators of a Sylow p -subgroup of the group. Clearly, if G is either p -solvable or a group with no p -solvable composition factors, then we do not recover, from the bound $l \leq 2d$ on the generalized p -length l of G , the above stated bounds, respectively on the p -length and on the non- p -solvable length. Although such a general bound for the generalized p -length cannot be obtained, we have not been able to set whether our estimation is the best possible of its kind (see Section 4).

2. TATE'S p -NILPOTENCY CRITERION AND A LEMMA

The proof of our Theorem A in the Introduction relies on Tate's p -nilpotency criterion for finite groups [4]. In this section, we briefly sketch a proof of Tate's well-known result with a small variation on the original arguments, and we also state some useful consequences of the proof.

Fix a prime p . As before, let G be a finite group and P a Sylow p -subgroup of G . Tate's criterion establishes that the restriction map

$$H^1(G) \longrightarrow H^1(P)$$

is a group isomorphism if and only if G has a normal p -complement. In order to prove this, let $N = O^p(G)$. In particular, note that N has no proper quotients of order a power of p , that is $O^p(N) = N$, and thus $H^1(N) = 0$. Write $M = N \cap P$, so the natural inclusion gives a natural isomorphism from P/M into G/N . Then we have the following commutative diagram, where all the arrows are natural:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & P/M \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N \longrightarrow 1. \end{array} \tag{1}$$

Now the inflation-restriction-transgression exact sequence in cohomology (see [5, Corollary 7.2.3]) leads to the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(G/N) & \xrightarrow{\alpha} & H^1(G) & \xrightarrow{\beta} & H^1(N)^G & \xrightarrow{\gamma} & H^2(G/N) & \xrightarrow{\delta} & H^2(G) \\
& & \downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_3 & & \downarrow \iota_4 & & \downarrow \iota_5 \\
0 & \longrightarrow & H^1(P/M) & \xrightarrow{\alpha'} & H^1(P) & \xrightarrow{\beta'} & H^1(M)^P & \xrightarrow{\gamma'} & H^2(P/M) & \xrightarrow{\delta'} & H^2(P).
\end{array} \quad (2)$$

Observe that since $H^1(N) = 0$, we have that δ is a monomorphism. Recall that ι_5 is a monomorphism (because P has index coprime to p in G) and ι_4 is an isomorphism, which implies that δ' is also a monomorphism. Therefore γ' is the null map and β' is surjective. On the other hand, note that both α and ι_1 are isomorphisms, and by hypothesis ι_2 is also an isomorphism. Hence α' is an isomorphism and β' is the null map.

Note that the two observations in the previous paragraph lead to the fact that $H^1(M)^P = 0$. But if M is non-trivial, since P is a p -group, then $H^1(M)^P$ is non-trivial. This implies that N is a normal p -complement of G , which is the conclusion of Tate's result.

In the above setting, if we drop the hypothesis that $H^1(G) \rightarrow H^1(P)$ is an isomorphism, we conclude the following, which can also be deduced from [4] or [6]:

Lemma 1. *Let G be a finite group and $P \in \text{Syl}_p(G)$. Suppose that N is a normal subgroup of G such that $N = O^p(N)$, and write $M = N \cap P$. Then the following sequence is exact:*

$$0 \longrightarrow H^1(P/M) \xrightarrow{\alpha'} H^1(P) \xrightarrow{\beta'} H^1(M)^P \longrightarrow 0. \quad (3)$$

Equivalently, the following sequence is exact:

$$0 \longrightarrow H_1(M)_P \xrightarrow{(\beta')^*} H_1(P) \xrightarrow{(\alpha')^*} H_1(P/M) \longrightarrow 0. \quad (4)$$

Proof. Consider the diagram (2) for the group $G = PN$. In that situation δ is injective and ι_4 is an isomorphism. This implies that γ' is the null map and the lemma follows. ■

The following well-known fact is a direct consequence of the previous lemma.

Corollary 2. *Let G be a finite group and P a Sylow p -subgroup of G . Suppose that N is a normal subgroup of G such that $P \cap N \leq \Phi(P)$. Then N is p -nilpotent.*

Proof. Apply the previous lemma to the group PN and its normal subgroup $O^p(N)$. By hypothesis $H^1(P/(P \cap O^p(N))) \cong H^1(P)$, and therefore $H^1(P \cap O^p(N))^P = 0$. Hence $P \cap O^p(N) = 1$, so $O^p(N)$ is a normal p -complement of N . ■

3. A p -SOLVABILITY CRITERION

In this section we work to prove Theorem A in the Introduction.

We start by proving that (1) implies (2) in Theorem A, which follows easily from Lemma 1. Let G be a finite group and $P \in \text{Syl}_p(G)$. Suppose first that G is p -solvable of p -length one, and write $M_1 = O^{p'}(G)$, $L_1 = O^p(M_1)$. Observe that since M_1 has a normal p -complement and $P \subseteq M_1$, it is clear that restriction defines an isomorphism:

$$f : H^1(M_1) \longrightarrow H^1(P).$$

Proposition 3. *Let G be a finite p -solvable group of p -length $l \geq 2$, and P a Sylow p -subgroup of G . Let $L_0 = G$, $M_i = O^{p'}(L_{i-1})$ and $L_i = O^p(M_i)$ for $i \geq 1$. Then the sequences*

$$0 \longrightarrow H^1(L_{j-1}P) \xrightarrow{f_j} H^1(L_jP) \xrightarrow{g_j} H^1(M_j)^P \longrightarrow 0,$$

where the maps f_j, g_j are restriction maps, are exact for $2 \leq j \leq l$. In particular, we have that $H^1(P) \cong \bigoplus_{i=1}^l H^1(M_i)^P$.

Proof. Since any normal subgroup of G of order coprime to p lies in the kernel of any homomorphism from G into \mathbb{F}_p , we can assume that G has no non-trivial normal p' -subgroups. We shall prove the result by induction on l .

Suppose first that G has p -length $l = 2$, and note that $M_1 = L_1P$ and $M_2 = P \cap L_1$ in this case. It is clear that $H^1(M_1) \cong H^1(P/M_2)$, so Lemma 1 implies that the sequence

$$0 \longrightarrow H^1(M_1) \xrightarrow{f_2} H^1(P) \xrightarrow{g_2} H^1(M_2)^P \longrightarrow 0$$

is exact, as wanted.

We now assume that the result holds for p -solvable groups of p -length at most $l-1 \geq 2$. Arguing as in the previous paragraph with the group $L_{l-1}P$ of p -length 2, we obtain that the sequence

$$0 \longrightarrow H^1(L_{l-1}P) \xrightarrow{f_l} H^1(P) \xrightarrow{g_l} H^1(M_l)^P \longrightarrow 0$$

is exact. Now, applying the inductive hypothesis on the group G/L_{l-1} we obtain $l-2$ exact sequences, which are easily seen to be the ones needed to complete the proof of the first statement of the proposition. The statement on the isomorphism of the first cohomology group of P follows directly from this. ■

Now it is easy to complete the proof of Theorem A.

Theorem 4. *Let G be a finite group, p a prime number and P a Sylow p -subgroup of G . Write $M_1 = O^{p'}(G)$ and $M_i = O^{p'}(O^p(M_{i-1}))$ for $i \geq 2$. Then the following two conditions are equivalent:*

- (1) G is p -solvable.
- (2) $H^1(P) \cong \bigoplus_{i \in \mathbb{N}} H^1(M_i)^P$.

Proof. By Proposition 3, it only remains to prove that (2) implies (1). Write $M_0 = G$, so $\{M_i\}_{i \geq 0}$ is a filtration of G that stabilizes at some point, say t :

$$M_t = M_{t+1} = M_{t+2} = \dots$$

Observe that G is p -solvable if and only if $M_t = 1$. By the choice of t , it is clear that $O^p(M_t) = M_t$. Therefore, by Lemma 1 the following sequence is exact:

$$0 \longrightarrow H^1(P/M_t \cap P) \longrightarrow H^1(P) \longrightarrow H^1(P \cap M_t)^P \longrightarrow 0.$$

By Proposition 3, condition (2) in the statement and the fact that $P/(M_t \cap P) \cong PM_t/M_t$ is a Sylow p -subgroup of the p -solvable group G/M_t , it follows that $H^1(P/M_t \cap P) \cong H^1(P)$. Thus $H^1(P \cap M_t)^P = 0$, which implies that $P \cap M_t = 1$, and this can only occur if $M_t = 1$. ■

4. GENERALIZED p -LENGTH AND p -PERFECT GROUPS

In this section we extend some ideas used to prove Theorem A, and give a proof of Theorem B in the Introduction. At the end of the section, we also propose a conjecture on a bound for the generalized p -length of a finite group.

Recall that a finite group G is perfect if $H_1(G, \mathbb{Z}) = 0$. Since $H_1(G, \mathbb{Z}_p)$ is the Sylow p -subgroup of $H_1(G, \mathbb{Z})$, it follows that G is perfect if and only if the homology group $H_1(G, \mathbb{Z}_p)$ is trivial for all primes p . A group satisfying any of the following equivalent properties is called a **p -perfect group**:

- (1) $H_1(G, \mathbb{Z}_p) = 0$,
- (2) $H_1(G, \mathbb{F}_p) = 0$,
- (3) $H^1(G, C_{p^\infty}) = 0$,
- (4) $H^1(G, \mathbb{F}_p) = 0$,
- (5) $O^p(G) = G$.

We say that a series of normal subgroups $\{N_i\}_{i=0}^r$ of G is a p -perfect filtration if $N_0 = G$, $N_r = 1$, $N_i \leq N_j$ for all $i \geq j$, and for all $i > 1$, the group N_i is p -perfect. We define the **p -perfect length** of $\{N_i\}_{i=0}^r$ as the number of factors N_i/N_{i+1} such that p divides $|N_{i+1} : N_i|$. We describe the main properties of p -perfect filtrations in the following proposition:

Proposition 5. *Let G be a finite p -group, P a Sylow p -subgroup and $\{N_i\}_{i=0}^r$ a p -perfect filtration. Write $M_i = N_i \cap P$ for all i . Then*

- (1) *For all $i \geq j$, the restriction map*

$$\text{res}_{M_i}^{M_j} : H_1(M_i)_P \longrightarrow H_1(M_j)_P$$

is injective, and it is an isomorphism if and only if p does not divide $|N_j : N_i|$.

- (2) *For all $i \geq j$, the restriction map*

$$\text{res}_{M_i}^{M_j} : H^1(M_j)^P \longrightarrow H^1(M_i)^P$$

is surjective, and it is an isomorphism if and only if p does not divide $|N_j : N_i|$.

Proof. Notice that the second statement follows directly from the first one by duality. Let us prove first the injectivity of the map on the homology groups. Since N_i is p -perfect, by Lemma 1 we have the following exact sequence

$$0 \longrightarrow H_1(M_i)_P \longrightarrow H_1(P) \longrightarrow H_1(P/M_i) \longrightarrow 0. \quad (5)$$

In particular, $H_1(M_i)_P \rightarrow H_1(P)$ is injective. Now, recall that the later arrows factor through $H_1(M_j)_P$. That is,

$$H_1(M_i)_P \longrightarrow H_1(M_j)_P \longrightarrow H_1(P). \quad (6)$$

This shows that the arrow $H_1(M_i)_P \rightarrow H_1(M_j)_P$ is injective.

Suppose now that $H_1(M_i)_P \rightarrow H_1(M_j)_P$ is an isomorphism. By Lemma 1 applied to the group N_j , we have the following exact sequence of P -modules:

$$0 \longrightarrow H_1(M_i)_{M_j} \longrightarrow H_1(M_j) \longrightarrow H_1(M_j/M_i) \longrightarrow 0. \quad (7)$$

Since taking co-invariants is right exact, the following sequence is exact:

$$H_1(M_i)_P \longrightarrow H_1(M_j)_P \longrightarrow H_1(M_j/M_i)_P \longrightarrow 0. \quad (8)$$

In particular, since $H_1(M_i)_P \rightarrow H_1(M_j)_P$ is an isomorphism, we have that $H_1(M_j/M_i)_P = 0$. This implies that $M_i = M_j$ and therefore p does not divide $|N_j : N_i|$. The converse is clear. ■

The following corollary is straightforward.

Corollary 6. *Let G be a finite p -group, P a Sylow p -subgroup of G and $\{N_i\}_{i=0}^r$ a p -perfect filtration. Then the p -perfect length of $\{N_i\}_{i=0}^r$ is at most the number of generators of P .*

Proof. Recall that the number of generators of P is the dimension of $H_1(P, \mathbb{F}_p)$. Then the corollary is clear from the previous proposition. ■

Next we present a natural way of constructing p -perfect filtrations.

Example 7. *Let G be a finite group and p a fixed prime. Write $N_0 = G$ and let M_1 be the p -solvable residue of G , that is M_1 is the smallest normal subgroup of G such that G/M_1 is p -solvable. Consider the normal series $\{N_j\}_{j=0}^{s_1}$ of G such that $N_0 = G$, $N_j = O^p(O^{p'}(N_{j-1}))$ for $j \geq 1$, and $N_{s_1} = M$. If $N_{s_1} \neq 1$, next we take N_{s_1+1} a normal subgroup of G such that N_{s_1}/N_{s_1+1} is a chief factor of G . Notice that by construction N_{s_1}/N_{s_1+1} is non- p -solvable, so in particular p divides $|N_{s_1} : N_{s_1+1}|$. Now we take M_2 the p -solvable residue of N_{s_1+1} . Observe that since M_2 is characteristic in N_{s_1+1} , it is normal in G . Then we can proceed to refine the filtration $N_{s_1+1} \supseteq M_2$ as above. In this way, and avoiding repetitions if they occur, we construct a filtration $\{N_j\}_{j=0}^r$, where $r \geq s_1$, with the following properties:*

1. *For all j , N_j/N_{j+1} is either a non- p -solvable chief factor of G or a p -nilpotent group.*
2. *For all j , if N_j/N_{j+1} is a non- p -solvable chief factor, then N_j is p -perfect.*
3. *For all j , if N_j/N_{j+1} is p -nilpotent, then N_{j+1} is p -perfect.*

In order to obtain the desired p -perfect filtration $\{J_i\}_{i=0}^t$ of G , we can just take the (ordered) subset of $\{N_j\}_{j=0}^r$ formed by the subgroups N_j such that either N_{j-1}/N_j is p -nilpotent, or N_j/N_{j+1} is non- p -solvable, together with $J_0 = G$ and the trivial subgroup $J_t = 1$. Of course, it is no loss to assume that we do not have repetitions in the series.

It is clear from the definition given in the Introduction that if G is p -solvable, then the p -length of G coincides with the generalized p -length of G . On the other hand, if G has no p -solvable chief factors then the generalized p -length of G is just the number of chief factors of G of order divisible by p that appear in any composition series of G .

Now we are ready to prove Theorem B.

Theorem 8. *Let G be a finite group and P a Sylow p -subgroup of G . Then the minimal number of generators of P is greater or equal to the p -length of G .*

Proof. Consider the filtration $\{J_i\}_{i=0}^t$ of G constructed in Example 7. The theorem now follows from Corollary 6, because the p -perfect p -length of $\{J_i\}_{i=0}^t$ is clearly greater or equal than the p -length of G . ■

It is also easy to deduce from Example 7 the following result due to Huppert for odd primes.

Theorem 9. *Let G be a finite group and P a Sylow p -subgroup of G . Then the minimal number of generators of P is greater or equal to the non- p -solvable p -length of G .*

After proving this two results, it arises the question whether one could combine both in a more general bound, that is, whether the number of generators of the Sylow p -subgroups bounds the generalize p -length of a finite group. Unfortunately this is not the case for the Schur cover of S_5 .

Example 10. *Let G be a Schur cover of S_5 . The generalized 2-length of G is 3, but the Sylow 2-subgroup of G is isomorphic to a Schur cover of D_8 , which can be generated by 2 elements.*

In any case, it seems that this situation is quite particular, and one should expect the following question to be true.

Question 11. *Is it true that for all, but a finite number of primes, the generalized p -length of a finite group is bounded by the number of generators of its Sylow p -subgroups?*

In order to give an answer to this question, and working with the filtration constructed in example 7, one needs to study sequences of G -groups:

$$1 \longrightarrow M \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1, \quad (9)$$

where H is a direct product of copies of a finite simple non-abelian group, M is a simple $\mathbb{F}_p[H]$ -module and G is a finite group into which \tilde{H} is embedded as a normal subgroup. Write Q for a Sylow p -subgroup of G and let $P = Q \cap \tilde{H}$. Under these circumstances, it would be interesting to know when at least one of the following two properties holds:

(1) The sequence

$$0 \longrightarrow H^1(P/M) \longrightarrow H^1(P) \longrightarrow H^1(M)^P \longrightarrow 0.$$

is exact.

(2) $\dim(H^1(P/M)^Q) \geq 2$.

For instance, if (9) is split, an easy argument shows that the transgression map $H^1(M)^P \rightarrow H^2(P/M)$ is the null map, and condition 1 holds.

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